

Inward matrix product algebra and calculus as tools to construct space–time frames of arbitrary dimensions

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In this study, inward matrix products are used to construct a theoretical framework where new space-time structures of arbitrary dimensions can be built up. The mathematical theory, based on inward matrix algebra, allows the derivation and integration of vectors and matrices composed by well-behaved functional elements. Every function element is associated at least to a linearly independent variable connected to such an element. As examples are discussed first the construction of general density functions, followed by the reformulation of the time-dependent Schrödinger equation. A general N -dimensional classical universe is presented, where not only space but also time, mass, energy and other related physical properties acquire an arbitrary hypermatrix structure. In this hypothetical framework scalar values related to physical quantities can be alternatively associated to cosine-like measures in the chosen spaces. Finally, simple problems on special relativity are briefly discussed from this point of view.

KEY WORDS: inward matrix product, inward matrix derivation and integration, hypermatrix space-time structures, N -dimensional time-dependent Schrödinger equation, classical mechanics in N -dimensional spaces, physical measures in N -dimensional spaces, special relativity in N -dimensional spaces

Introduction

Inward matrix products (IMP) have been studied and used in several applications involving some aspects of quantum chemistry [1] and quantum density functions [2] in general, as well as have been employed to discuss quantum similarity and quantum QSAR in particular [3]. The essential IMP features are already implemented in the standard Fortran 95 compilers, see, for example, [4], thus the immediate use of IMP in computational structures is assured, as was already commented in previous work [5–7]. It is most curious that the IMP is so barely present in the literature associated to the usual theoretical or computational chemical environment and has been almost unnoticed in mathematical discussions. References to IMP, as far as the author is aware, are scarce as the following encyclopaedic quotation shows [8]. Thus, the interest into developing IMP properties and applications cannot only have physical and chemical interest but even can be found to possess a certain applied mathematical point of view utility.

IMP algebraic and functional definitions and their use have already been provided in several places [1,3,5–7]. IMP has been employed to obtain the associated matrix powers and functions and has been described as an alternative way leading to matrix symmetrization [5]. Use of IMP to obtain constrained approximate solutions of linear equations has been also reported and used in a quantum QSAR context [3] from several points of view [5,7], mainly related to the relationship of QSAR with the so-called quantum similarity theory (see for recent reviews [9–13]). Such continued application possibilities have triggered the first steps of the present report.

After reviewing general features of IMP algebra, the definitions of possible inward matrix derivatives and integrals will be given, leading to set up the main background in order to permit the construction of the body of an IMP-like calculus. As the purpose of this paper is to apply IMP calculus to various physical domains; thus, some applications of the whole computational structure will follow, among them, there are the construction of general density functions and a particular form of some special kind of time-dependent Schrödinger equation, which will be presented in a (hyper-)matrix formalism. Finally, it is presented a scheme of the plausible description of a symmetric space-time of arbitrary dimensions. In this hypothetical situation, the usual location and properties of physical objects can be arranged to adopt a structure of (hyper-)matrix. In this circumstances, scalar physical observables can be supposedly measured by means of the cosine of the angle subtended by two vectors, defined in turn within the physical object (hyper-)matrix spaces. Both classical and special relativity scenarios are briefly discussed in this context.

1. Inward matrix product (IMP)

As an introduction to IMP and in order to spare the potential reader the time to study other references, a brief description of the properties of IMP will be provided here. In this sense, the following definition can serve to build a necessary starting practical IMP structure. However, IMP has been defined in various ways and with several purposes, more information about these aspects can be found within [1,3,7,14,15].

Definition 1 (Inward (hyper-)matrix¹ product). Consider any arbitrary (hyper-)matrix space over a field \mathbf{K} : $\mathbf{M}_{(\times n)}(\mathbf{K})$. Let $\mathbf{A}, \mathbf{B} \in \mathbf{M}_{(\times n)}$. An IMP involving the known (hyper-)matrix pair is a closed operation, resulting in a new (hyper-)matrix $\mathbf{P} \in \mathbf{M}_{(\times n)}$, and symbolised by $\mathbf{P} = \mathbf{A} * \mathbf{B}$, whose elements are defined by the algorithm:

$$\forall(\mathbf{i}): \quad p(\mathbf{i}) = a(\mathbf{i})b(\mathbf{i}).$$

Above, the elements of the involved (hyper-)matrices are identified by means of an index vector $\mathbf{i} \equiv (i_1, i_2, \dots, i_p)$. Thus, the (hyper-)matrix space dimension is given by

¹ Along this study the term (hyper-)matrix will be used in order to note the feature consisting in that every given result or definition can be employed both in matrices or in a much more general framework, made by hypermatrices of arbitrary dimension. Such a notation has been already employed for similar purposes [7].

$(\times \mathbf{n}) \equiv (n_1 \times n_2 \times \cdots \times n_p)$. The notation follows a previous one, employed when dealing with nested sum structures [9–13] and generalising in this manner any (hyper-) matrix form.

1.1. IMP general features

1.1.1. IMP properties

The interest in defining such an IMP stems from the possibility to attach to it the most usual features of a multiplication composition rule, resembling the usual scalar algebra features, whilst defined in (hyper-)matrix spaces of arbitrary dimensionality.

The following characteristic properties can be attached to the IMP defined over the elements of a (hyper-)matrix space $\mathbf{M}_{(\times \mathbf{n})}$: IMP is distributive with respect (hyper-) matrix sum, as well as associative, and commutative [1]. It is straightforward to demonstrate that from definition 1, these commented properties can be associated to IMP, and thus it will not be repeated here. Under this simple definition of IMP, matrices can thus be allowed to behave almost as scalars. IMP properties have been described several times in the literature [1,5–7].

1.1.2. IMP unit element and inverse

A. Unity (hyper-)matrix as IMP neutral element. An IMP unit element exists, which can be called the *unity* (hyper-)matrix, and represented by a bold unit symbol, that is, $\mathbf{1} \in \mathbf{M}_{(\times \mathbf{n})}$, such that under IMP

$$\mathbf{1} * \mathbf{A} = \mathbf{A} * \mathbf{1} = \mathbf{A}.$$

Using the real multiplication unit, the unity (hyper-)matrix elements can be defined by means of

$$\mathbf{1} = \{1(\mathbf{i}) = 1 \forall(\mathbf{i})\}. \quad (1)$$

B. IMP inverse (hyper-)matrix. A new (hyper-)matrix defines the IMP inverse of a matrix \mathbf{A} , which can be noted as

$$\mathbf{A}^{[-1]} = \{a^{[-1]}(\mathbf{i})\},$$

with elements, which can be computed as follows:

$$\forall(\mathbf{i}): a^{[-1]}(\mathbf{i}) = (a(\mathbf{i}))^{-1}.$$

This definition produces the sequence of equalities under IMP:

$$\mathbf{A} * \mathbf{A}^{[-1]} = \mathbf{A}^{[-1]} * \mathbf{A} = \mathbf{1}.$$

The existence of an IMP inverse is subject to the following important limitation, only if the additional property holds:

$$\mathbf{A} = \{a(\mathbf{i}) \wedge \forall(\mathbf{i}): a(\mathbf{i}) \neq 0\}, \quad (2)$$

then the (hyper-)matrix \mathbf{A} can be called either IMP invertible, IMP regular or IMP non-singular. A (hyper-)matrix will be called IMP singular if condition (2) appears to be false.

C. Hadamard algebra. These properties are sufficient to define a commutative algebra over any (hyper-)matrix vector space. One can refer to this kind of algebra as *Hadamard algebra*.

1.1.3. IMP powers and functions

IMP powers of a given (hyper-)matrix \mathbf{A} are readily defined as $\mathbf{A}^{[p]} = \{a(\mathbf{i})^p\}$. The square brackets enveloping the exponent is used here, to distinguish an IMP power from the one defined involving classical matrix products, in the same way as such notation was employed in the IMP inverse definition. For example, whenever $\mathbf{Z} = \mathbf{A} * \mathbf{A}$, then the (hyper-)matrix \mathbf{A} can be also considered as the IMP square root of \mathbf{Z} , that is:

$$\mathbf{A} = \mathbf{Z}^{[1/2]} \longrightarrow \forall(\mathbf{i}): a(\mathbf{i}) = \sqrt{z(\mathbf{i})}.$$

IMP functions of a given (hyper-)matrix are also easy to define. For example, in general one can use

$$\phi[\mathbf{Z}] = \{\phi(z(\mathbf{i}))\}.$$

As noted in several previous papers [3,5,7], IMP algebra is tightly related to diagonal matrix products computational algorithms, and the above definitions are the consequence of another shared isomorphic characteristic between both sets.

1.2. IMP applications

Having described the main features of the IMP, two application examples will be given, in order that the reader can grasp the interest of such a simple IMP computational structure. Such examples have been chosen among other possible ones, as they will be employed in later discussion within this paper.

1.2.1. Scalar product of two (hyper-)matrices

The IMP involving two (hyper-)matrices can be trivially related to scalar products. If such a scalar product is defined, using a nested summation symbol (see, e.g., [16–19]) as

$$\langle \mathbf{A} | \mathbf{B} \rangle = \sum(\mathbf{i}) a(\mathbf{i})b(\mathbf{i}).$$

then one can also employ the auxiliary definition [16–19]

$$\langle \mathbf{A} \rangle = \sum(\mathbf{i}) a(\mathbf{i}),$$

in order to easily symbolise the sum of *all* elements of a given (hyper-)matrix, which could be called a *complete matrix elements summation* operator. This linear operator summation device is also a Fortran 95 compiler intrinsic feature, see for more details [4],

or any compiler manual. The linearity of the complete matrix elements summation operator can be straightforwardly seen as follows:

$$\begin{aligned} \langle \alpha \mathbf{A} + \beta \mathbf{B} \rangle &= \sum(\mathbf{i}) [\alpha a(\mathbf{i}) + \beta b(\mathbf{i})] \\ &= \alpha \sum(\mathbf{i}) [a(\mathbf{i})] + \beta \sum(\mathbf{i}) [b(\mathbf{i})] \\ &= \alpha \langle \mathbf{A} \rangle + \beta \langle \mathbf{B} \rangle. \end{aligned}$$

Then, taking into account the previous IMP definition and the complete matrix elements summation operator details, the next equality might be immediately written:

$$\langle \mathbf{A} | \mathbf{B} \rangle = \langle \mathbf{A} * \mathbf{B} \rangle.$$

1.2.2. Scalar product example

As an application example of the previous scalar product definition, a special property of quadratic forms can be studied. A quadratic form involving the (hyper-)matrix $\mathbf{A} = \{a(\mathbf{i}, \mathbf{j})\}$ as well as an arbitrary (hyper-)vector, $\mathbf{x} = \{x(\mathbf{i})\}$, may be written as

$$q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum(\mathbf{i}) \sum(\mathbf{j}) a(\mathbf{i}, \mathbf{j}) x(\mathbf{i}) x(\mathbf{j}). \quad (3)$$

It is well known that if \mathbf{A} is positive definite, then $\forall \mathbf{x} \neq \mathbf{0} \rightarrow q(\mathbf{x}) > 0$. The quadratic form (3) can be also written using the tensorial product of the variable (hyper-)vector, employing the algorithm: $\mathbf{T} = \mathbf{x} \otimes \mathbf{x} = \{t(\mathbf{i}, \mathbf{j}) = x(\mathbf{i})x(\mathbf{j})\}$. Then, the following IMP construction of the quadratic form will be also possible:

$$q(\mathbf{x}) = \langle \mathbf{A} * \mathbf{T} \rangle = \sum(\mathbf{i}) \sum(\mathbf{j}) a(\mathbf{i}, \mathbf{j}) t(\mathbf{i}, \mathbf{j}).$$

Define now the *signature* of a (hyper-)matrix as another binary (hyper-)matrix where the binary digits, taken as elements of the signature (hyper-)matrix, are in accordance with the signs of the (hyper-)matrix elements [3], that is, for example:

$$\mathbf{S}_A = \text{Sign}(\mathbf{A}) = \{s_A(\mathbf{i}) = \delta[a(\mathbf{i}) > 0]\},$$

where a *logical Kronecker's delta* has been also used [16–19] in the signature definition. A logical Kronecker's delta symbol is a generalization of the well-known one, which is defined through a logical expression L . The symbol $\delta[L]$ has the values $\{0, 1\}$ when the logical expression argument takes the values $\{F, T\}$, respectively. In the particular case of the matrix signature under the IMP, the corresponding bit product shall conform to the sign product rules, that is:

$$\{0 \times 0 \equiv 1 \times 1 = 1 \wedge 1 \times 0 \equiv 0 \times 1 = 0\}.$$

This can be associated to the meanings: $0 \equiv -1 \wedge 1 \equiv +1$.

As an application of the (hyper-)matrix signature discussed above, if a known (hyper-)matrix structure \mathbf{A} , whenever $\text{Sign}(\mathbf{A}) \neq \mathbf{1}$ holds, then the following (hyper-)matrix, the definition of the IMP absolute value of \mathbf{X} , can be easily set, as then any (hyper-)matrix can be decomposed in the following way: $*|\mathbf{A}| = \text{Sign}(\mathbf{A}) * \mathbf{A}$, which is

such that $\text{Sign}(*|\mathbf{A}|) = \mathbf{1}$. So, with this in mind and employing the auxiliary definition of the IMP absolute value function, one can write

$$*|\mathbf{A}| = \{ |a(\mathbf{i})| \forall(\mathbf{i}) \}. \quad (4)$$

Consequently, whenever

$$\text{Sign}(\mathbf{A}) = \text{Sign}(\mathbf{T}) \rightarrow \text{Sign}(\mathbf{A} * \mathbf{T}) = \mathbf{1}$$

hold, then, in this particular circumstance, it is always assured that $q(\mathbf{x}) > 0$. It can be concluded that in quadratic form evaluation, when a variable (hyper-)vector \mathbf{x} is chosen such that the following signature relationship holds: $\text{Sign}(\mathbf{x}) \otimes \text{Sign}(\mathbf{x}) = \text{Sign}(\mathbf{A})$, this will provide a positive value of the quadratic form.

When a unity signature is everywhere present in the studied (hyper-)matrix \mathbf{A} , all the considered elements are real and positive. In this special case, one can use the symbol $\mathbf{A} * > 0$, to indicate this property, alternatively: $\text{Sign}(\mathbf{A}) = \mathbf{1}$. The quadratic form positive values $q(\mathbf{x}) > 0$ will be always assured, when a signature $\mathbf{1}$ is chosen to hold in the variable (hyper-)vector $\mathbf{x} * > 0$. This is the same to say that quadratic forms are positive definite within a *vector semispace* (VSS)², see [20–24] for more details on VSS definition and characteristic features.

1.2.3. IMP and Taylor series expansions of multivariate functions

A Taylor series of a multivariate function, $f(\mathbf{x})$, in the neighborhood of a point \mathbf{x}_0 could be expressed in a simple form using the IMP of two tensors, as

$$f(\mathbf{x}) = \sum_{p=0}^{\infty} \langle \partial_p [f(\mathbf{x}_0)] * \otimes_p [\mathbf{x} - \mathbf{x}_0] \rangle,$$

where the symbol $\partial_p [f(\mathbf{x}_0)]$ collects all the p th order partial derivatives of the function evaluated at $\mathbf{x} = \mathbf{x}_0$; and $\otimes_p [\mathbf{x} - \mathbf{x}_0]$, collects the elements of the p th order tensor product of the vector difference argument. It must be noted that $\partial_0 [f(\mathbf{x}_0)] = f(\mathbf{x}_0)$ and $\otimes_0 [\mathbf{x} - \mathbf{x}_0] = 1$.

2. Inward matrix derivation and integration

Let us describe the basic operations to construct the rudiments of IMP calculus.

² A VSS is defined as a vector space, where the addition possesses a semigroup structure. All the involved scalars belong to \mathbb{R}^+ . No reciprocal vectors are present, so no vector differences appear whatsoever, when developing the appropriate algebra.

2.1. Parameterised (hyper-)matrices

Suppose a (hyper-)matrix made of well-behaved functional elements, whose variables constitute, each of them, an independent item. This is the same as to construct the adequate definition of a (hyper-)matrix of functions, by means of

$$\mathbf{Z}[\mathbf{T}] = \{z(\mathbf{i})[t(\mathbf{i})]\}. \quad (5)$$

Square brackets have been used in the formula above to denote the kind of functional of independent variables, which will be employed here. In the same manner, the (hyper-)matrix \mathbf{T} elements can freely be thought as formed, in turn, of vectors or matrices instead of scalars, although this will complicate unnecessarily the following analysis and will not be considered for the moment.

The usual simplest form of such functional dependence can be readily expressed as

$$\mathbf{T} = t\mathbf{1}, \quad (6)$$

where t represents a unique variable or parameter, which becomes common to every (hyper-)matrix functional element and the auxiliary unity (hyper-)matrix $\mathbf{1}$ has been already defined in equation (1).

The (hyper-)matrix \mathbf{T} will be called from now on a *parameter* (hyper-)matrix and any (hyper-)matrix structure like $\mathbf{Z}[\mathbf{T}]$ will be a *parameterised* (hyper-)matrix. Note that if the property $\mathbf{Z}[\mathbf{T}] * > 0$ has to be considered to hold, then the set of functions forming the elements of the parameterised (hyper-)matrix shall be positive definite, that is, $\forall(\mathbf{i}): z(\mathbf{i})[t(\mathbf{i})] > 0$.

2.2. Parameterised (hyper-)matrices IMP derivatives and integrals

IMP derivatives and integrals could be defined over parameterised (hyper-)matrices in a very particular way, conforming to the structure and properties of the IMP operations, discussed in the previous section. IMP calculus will become an interesting tool in order to generalize usual physical concepts related to both classical and quantum mechanics.

2.2.1. IMP derivatives

The IMP derivative of a parameterised (hyper-)matrix can be defined as in the straightforward one-parameter case, by means of a symbolic form with a well-defined meaning, similar to the one-parameter function matrix derivatives. The IMP (hyper-)matrix derivative has to be considered a way to derive every parameterised (hyper-)matrix element with respect to the associated independent variable. That is:

$$* \frac{d}{d\mathbf{T}}(\mathbf{Z}[\mathbf{T}]) = \left\{ \frac{d}{dt(\mathbf{i})}(z(\mathbf{i})[t(\mathbf{i})]) \right\}. \quad (7)$$

Of course, when the (hyper-)matrix possess a functional structure like the one depicted in equation (6), then the IMP derivative (7) transforms into a conventional matrix derivative:

$$*\frac{d}{d\mathbf{T}}(\mathbf{Z}[t\mathbf{1}]) = \left\{ \frac{d}{dt}(z(\mathbf{i})[t]) \right\}. \quad (8)$$

Higher derivatives are obtained by applying rule (7) as many times as needed. For example, considering the case of the equation (8), the p th order IMP derivative can be written as

$$*\frac{d^p}{d\mathbf{T}^p}(\mathbf{Z}[t\mathbf{1}]) = \left\{ \frac{d^p}{dt^p}(z(\mathbf{i})[t]) \right\}.$$

2.2.2. IMP integration

The IMP integration rules follow the same path as the previous IMP derivatives described before, and in this manner an IMP integral of a parameterised matrix is readily and coherently constructed as

$$*\int_{\Omega} [\mathbf{Z}[\mathbf{T}]] d\mathbf{T} = \left\{ \int_{\Omega} z(\mathbf{i})[t(\mathbf{i})] dt(\mathbf{i}) \right\}.$$

2.2.3. Simple illustrative examples of IMP calculus

In order to illustrate the definitions presented above, examples of IMP derivation and integration of parameterised matrices will be discussed next.

A. IMP derivative of a parameterised vector. Suppose an N -dimensional vector space, where every vector element can be described by means of a continuous function of an independent variable, admitting derivatives up to an arbitrary order. Then

$$\forall \mathbf{x}(\mathbf{t}) \in \mathcal{V}_n \wedge \mathbf{t} \in \mathcal{C}_n(\mathbf{R}) \quad \longrightarrow \quad \mathbf{x}(\mathbf{t}) = \{x_i(t_i)\}. \quad (9)$$

Thus, the IMP derivative of such an N -dimensional vector can be defined as follows:

$$*\frac{d}{d\mathbf{t}}[\mathbf{x}(\mathbf{t})] = \left\{ \frac{d}{dt_i}x_i(t_i) \right\}.$$

Suppose that, moreover, the parameterised vectors $\mathbf{x}(\mathbf{t})$ are made of polynomials of arbitrary order in the independent variables, that is:

$$\mathbf{x}(\mathbf{t}) = \left\{ x_i(t_i) = \sum_v \gamma_{vi} t_i^v \right\}, \quad (10)$$

then the IMP derivative elements will be written

$$*\frac{d}{d\mathbf{t}}[\mathbf{x}(\mathbf{t})] = \left\{ \frac{d}{dt_i}x_i(t_i) = \sum_v v\gamma_{vi} t_i^{v-1} \right\}.$$

B. IMP integral of a parameterised vector. Using the same kind of parameterised vectors as those of the previous example and defined in equation (9), the IMP integral of such a vector is defined in turn as

$$* \int_{\Omega} \mathbf{x}(\mathbf{t}) \, d\mathbf{t} = \left\{ \int_{\Omega} x_i(t_i) \, dt_i \right\}.$$

In case that the polynomial structure (10) is chosen and the integration domain admittedly adopted as $[0, +1]$, then one can write

$$* \int_{\Omega} \mathbf{x}(\mathbf{t}) \, d\mathbf{t} = \left\{ \int_0^1 x_i(t_i) \, dt_i = \sum_v v^{-1} \gamma_{vi} \right\}.$$

2.3. Generalised density functions

The IMP features permit one to express density functions in a compact and extremely general form. Previous work [7] may be invoked to define a (hyper-)matrix \mathbf{W} as a convex coefficient (hyper-)matrix of arbitrary dimensions ($\times \mathbf{n}$). Convexity meaning has to be assumed here as, besides the property $\langle \mathbf{W} \rangle = 1$, also that all elements of \mathbf{W} are real and positive, that is: $\mathbf{W} * > 0$.

Such a (hyper-)matrix with unity signature can be formed by choosing a complex (hyper-)matrix \mathbf{X} , with the same shape and dimension as \mathbf{W} , such that $\mathbf{W} = \mathbf{X}^* * \mathbf{X}$. In this way, it can be assured that

$$\mathbf{W} = \{ \forall \omega(\mathbf{i}) \in \mathbf{W} \rightarrow \exists \chi(\mathbf{i}) \in \mathbf{X}: \omega(\mathbf{i}) = \chi(\mathbf{i})^* \chi(\mathbf{i}) = |\chi(\mathbf{i})|^2 \in \mathbb{R}^+ \}.$$

Such special choice can be also described by the *convex conditions* symbol on \mathbf{W} [20–23], written as

$$K_{\{\times \mathbf{n}\}}(\mathbf{W}) = \{ \forall \omega(\mathbf{i}) \in \mathbf{W} \rightarrow \omega(\mathbf{i}) \in \mathbb{R}^+ \wedge \langle \mathbf{W} \rangle = 1 \},$$

or in a shorthand way:

$$K_{\{\times \mathbf{n}\}}(\mathbf{W}) = \{ \mathbf{W} * > 0 \wedge \langle \mathbf{W} \rangle = 1 \}.$$

Define now a (hyper-)matrix \mathbf{P} with a shape and dimension equal to the one of \mathbf{W} , and containing as elements, normalised positive definite multivariate functions of a variable (hyper-)matrix \mathbf{t} of arbitrary dimension. It can be formally written:

$$* \int \mathbf{P}(\mathbf{t}) \, d\mathbf{t} = \mathbf{1},$$

where $\mathbf{1}$ is a unity (hyper-)matrix of the appropriate shape and dimension too, that is: $\text{Dim}(\mathbf{1}) = \text{Dim}(\mathbf{P})$. More specifically, one can compactly build up the convenient structure of the (hyper-)matrix \mathbf{P} as

$$\mathbf{P} = \left\{ \forall p(\mathbf{i})[t(\mathbf{i})] \in \mathbf{P} \rightarrow \forall t(\mathbf{i}): p(\mathbf{i})[t(\mathbf{i})] \in \mathbb{R}^+ \wedge \int p(\mathbf{i})[t(\mathbf{i})] \, dt(\mathbf{i}) = 1 \right\}.$$

This can be accomplished by constructing an IMP using a (hyper-)matrix with elements made of functions Φ with the appropriate shape and dimension as

$$\mathbf{P} = \Phi^* * \Phi,$$

so, in this manner the (hyper-)matrix \mathbf{P} can be written as

$$\mathbf{P} = \{\forall p(\mathbf{i}) \in \mathbf{P} \rightarrow \exists \varphi(\mathbf{i}) \in \Phi: p(\mathbf{i}) = |\varphi(\mathbf{i})|^2\}.$$

A normalised density function of the variable (hyper-)matrix \mathbf{t} can be expressed as an IMP manipulation producing a scalar product:

$$\begin{aligned} \rho(\mathbf{t}) = \langle \mathbf{W} * \mathbf{P}(\mathbf{t}) \rangle &\Rightarrow \int \rho(\mathbf{t}) \, d\mathbf{t} = \int \langle \mathbf{W} * \mathbf{P}(\mathbf{t}) \rangle \, d\mathbf{t} \\ &= \left\langle \mathbf{W} * \int \mathbf{P}(\mathbf{t}) \, d\mathbf{t} \right\rangle \\ &= \langle \mathbf{W} * \mathbf{1} \rangle = \langle \mathbf{W} \rangle = 1. \end{aligned}$$

3. Time-dependent Schrödinger equation

The well-known time-dependence of Schrödinger classical wave functions can be generalised into a time-like parameterised (hyper-)vector or matrix. To analyse this possibility, first the scalar time parameterised Schrödinger equation can be explicitly written in atomic units as (see, e.g., [25,26])

$$\mathbf{H}\Phi(\mathbf{R}, t) = i \frac{\partial \Phi(\mathbf{R}, t)}{\partial t}. \quad (11)$$

There the usual meaning has to be taken for the symbols employed: \mathbf{H} is the system's Hamiltonian operator, $\Phi(\mathbf{R}, t)$ the time-dependent wave function, with \mathbf{R} denoting the system's particles position coordinates in matrix form, and t the scalar time. The usual consistent form [25,26] of the time-dependent wave function is

$$\Phi(\mathbf{R}, t) = \Psi(\mathbf{R}) \exp(-iEt), \quad (12)$$

where E is the system's state energy.

From this point of view, nothing opposes to the fact that the wave function can be considered parameterised by a time-like (hyper-)matrix taking symbolically in this case the form $\Phi(\mathbf{R}, \mathbf{T})$. The factorisation (12) could be structured in terms of an IMP exponential function like

$$\Phi(\mathbf{R}, \mathbf{T}) = \Psi(\mathbf{R}) \exp(-iE\mathbf{T}), \quad (13)$$

where the exponential part of the decomposition (13) may be defined in turn as another IMP exponential function:

$$\mathbf{Z}[\mathbf{T}] = \exp(-iE\mathbf{T}) = \{z(\mathbf{i}) = \exp(-iET(\mathbf{i}))\},$$

so in this way the time-dependent Schrödinger equation equivalent to equation (11) may be written using the IMP derivative concept, as defined in section 2.2.1:

$$H\Phi(\mathbf{R}, \mathbf{T}) = i \left[* \frac{d}{d\mathbf{T}} \Phi(\mathbf{R}, \mathbf{T}) \right]. \quad (14)$$

So, when a decomposition of type (13) is performed, then the IMP derivative in equation (14) can be written as

$$i \left[* \frac{d}{d\mathbf{T}} \Phi(\mathbf{R}, \mathbf{T}) \right] = E \Phi(\mathbf{R}, \mathbf{T}).$$

In this way it is obvious that equation (14), like the classical counterpart in equation (11), can be equally transformed into the time-independent Schrödinger equation

$$H\Psi(\mathbf{R}) = E\Psi(\mathbf{R}).$$

It is also straightforward to see how the time-dependent perturbation theory can be easily set in the parameterised time-like (hyper-)matrix wave function scheme in the same way as in the scalar framework.

4. Parameterised (hyper-)matrix structure of classical space–time

The naive exercise performed over the classical Schrödinger equation, as in section 3 above, may indicate that even at macroscopic classical or relativistic mechanics, it is possible to use a parameterised (hyper-)matrix structure of space–time. If this prospect is feasible, then the usual scalar results, related to the experimental values of physical measurement, shall have necessarily a connection with the N -dimensional nature of the space–time structure, compatible with the experimental evidence of the usual scalar results. In this section one among other possible solutions of this general framework will be studied.

4.1. Velocity, momentum and mass

Suppose a symmetric space–time with respect to the dimensions of both space and time, made of (hyper-)matrix coordinates. That is: the dimension of space and time coordinates will be the same for a given physical object description. Moreover, suppose that some Euclidean space–time structure holds too. Then, the space coordinates $\mathbf{Z}[\mathbf{T}]$ can be built up as a parameterised (hyper-)matrix, in the same manner as in equation (5). A velocity (hyper-)matrix can be then written as the IMP derivative:

$$\mathbf{V}[\mathbf{T}] = * \frac{d}{d\mathbf{T}} \mathbf{Z}[\mathbf{T}]. \quad (15)$$

Thus, the momentum can be defined using a (hyper-)matrix mass structure of the appropriate dimensions, forming a (hyper-)matrix structure, also of the same dimensions as space and time coordinates, employing the IMP, in the following fashion:

$$\mathbf{P}[\mathbf{T}] = \mathbf{M} * \mathbf{V}[\mathbf{T}]. \quad (16)$$

Classically time and mass are positive definite scalar quantities, thus, apparently a problem arises on how to connect this uncommon (hyper-)matrix panorama with the customary time and mass scalar measures. This situation shall be clarified now, before going further away in the theoretical development.

To start with, one can consider that time and mass being positive definite in our custom world, even if they become associated to a (hyper-)matrix structure, then their elements could be considered restricted to be positive. Thus, the (hyper-)matrices \mathbf{M} and \mathbf{T} can be, in this way, considered as elements of a vector semispace with the appropriate dimension. As it has been defined: $\mathbf{M} * > 0 \wedge \mathbf{T} * > 0$.

4.2. Scalar measures

From here, one can consider the characteristics of vector semispaces being of such a nature that, as not allowing negative vector elements nor scalars, then Euclidian-like distance measures are not allowed too, thus, a plausible remnant solution, in order to measure the mass-like and time-like (hyper-)matrices, while transforming them into scalar positive definite modules, is the inverse of a distance measure: a cosine of the angle between two vectors.

Then, it is necessary now to describe in a (hyper-)matrix context how a cosine-like scalar can be computed and how this result can be associated to another scalar with the appropriate range of a distance, if needed. In order to see the general procedure suppose that a unity signature (hyper-)matrix \mathbf{Z} , $\mathbf{Z} * > 0$, shall be compared against another reference (hyper-)matrix \mathbf{Z}_0 , say, with the same unit signature characteristics: $\mathbf{Z}_0 * > 0$. First, the following positive definite function can be computed:

$$\gamma(\mathbf{Z}; \mathbf{Z}_0) = \langle \mathbf{Z} * \mathbf{Z}_0 \rangle (\langle \mathbf{Z} * \mathbf{Z} \rangle \langle \mathbf{Z}_0 * \mathbf{Z}_0 \rangle)^{-1/2}. \quad (17)$$

Such a function yields a range of values included in the interval $\gamma \in [0, +1]$. The reciprocal positive definite measure $g = \gamma^{-1} - 1$ produces a new distance-like range, as now one will have $g \in [+∞, 0]$, and consequently, one can consider this distance-like parameter as a real non-negative scalar.

To see this applied to the previously defined mechanical (hyper-)matrices, suppose that time-like (hyper-)matrices wait to be measured, with respect of a time-like (hyper-)matrix \mathbf{T}_0 , which acts as a reference. The appropriate scalar measure, associated to the actual time-like (hyper-)matrix \mathbf{T} , will be easily based, using equation (17), on the cosine expression:

$$\tau = \gamma(\mathbf{T}; \mathbf{T}_0). \quad (18)$$

Then, the scalar time measure $t = \tau^{-1} - 1$, will, thus, behave as a distance-like form, providing a positive definite scalar time measure. This is so because the equation (18) is nothing else than the cosine of the angle subtended between two time (hyper-)matrix elements of a VSS: \mathbf{T} and \mathbf{T}_0 . Because it represents the cosine of two unity signature (hyper-)matrices, it will possess a range of values defined like $\tau \in [0, +1]$, thus, the scalar time measure t will have the positive definite range $t \in [+∞, 0]$, corresponding to the former variation limits of τ .

The same will occur between the mass-like (hyper-)matrices \mathbf{M} , when compared against a convenient reference (hyper-)matrix \mathbf{M}_0 . A cosine like measure $\mu = \gamma(\mathbf{M}; \mathbf{M}_0)$, defined exactly as in equation (17), will produce the desired result. So, in this way a positive definite scalar mass $m = \mu^{-1} - 1$ can be defined too, ranging in the complete possible interval of positive real numbers.

Of course, this measuring protocol, set up in order to convert (hyper-)matrix objects into scalars, resembles the usual physical measures of length, time and mass, by using an object of the same nature as a measuring standard and yielding a scalar value as a result.

4.3. Kinetic energy

Continuing in this path, kinetic energy can be also structured as a (hyper-)matrix \mathbf{K} , by employing the structures already defined, thus, one can write

$$\mathbf{K} = \frac{1}{2} \mathbf{M} * \mathbf{V}^{[2]}, \tag{19}$$

where the velocity (hyper-)matrix, as defined in equation (15), is squared in an IMP power definition:

$$\mathbf{V}^{[2]} = \mathbf{V} * \mathbf{V}.$$

The interesting fact consists here in that the kinetic energy structure in the form of a (hyper-)matrix can be also considered as having unity signature, or $\text{Sign}(\mathbf{K}) = \mathbf{1}$. Thus, the scalar measure of this mechanical object characteristic can be handled in the same way, as has been commented in the time-like and mass-like cases.

So, a scalar kinetic energy measure can be defined using a reference (hyper-)matrix \mathbf{K}_0 , by means of computing the cosine-like function $\kappa = \gamma(\mathbf{K}; \mathbf{K}_0)$, and then a straightforward distance-like form $k = \kappa^{-1} - 1$.

Also, as it is usual, the kinetic energy expression (19) can be expressed in terms of the momentum structure (16), as the IMP products and powers can be used accordingly:

$$\mathbf{K} = \frac{1}{2} \mathbf{M}^{[-1]} * \mathbf{P}^{[2]}. \tag{20}$$

In the light of these definitions, an interesting property appears to be necessarily associated now to the mass-like (hyper-)matrices. As the IMP inverse of the mass (hyper-)matrix has to be available, for example, at least to compute momentum expressions like equation (20), then the mass-like (hyper-)matrices cannot possess any element null: they

shall be defined as (hyper-)matrices with strictly positive definite non-null elements. Thus, if mass is constructed in a symmetrically dimensional space–time structure as a (hyper-)matrix of the same dimensions as the space–time coordinates possess, in order that this can be possible and that no impediments to construct other IMP functions of mass-like objects shall arise, these mathematical structures attached to the physical objects could only bear strictly positive elements.

4.4. Some remarks

Therefore, in this way, one can see that a classical physical universe can be described in a coherent space–time, dimensionally symmetric, but also constructed with arbitrary dimensions. The scalar measures of distance, time, mass and kinetic energy can be obtained as the result of a cosine-like similarity measure, comparing the actual N -dimensional structure with a reference (hyper-)matrix of the appropriate dimensions. For example, in case that one chooses a tri-dimensional space structure and a symmetric tri-dimensional time structure, mass vectors will be tri-dimensional too, and so on.

5. Parameterised symmetric space–time in a special relativity framework

As it has been discussed in the previous section, the classical naive description structure of common mechanical and its most relevant computational elements can be coherently described in a completely general, dimensionally symmetric, classical space–time frameworks. Thus, there seems that no problems have to be present when entering the special relativistic parent point of view, and thus, similar results could be expected, if instead of a Euclidean space, an appropriate Minkowski space is considered as the background space–time frame. A symmetric Minkowski space–time is straightforward to construct, and its definition will not proceed any further here. Of course, now the interest of the discussion shall be focused in this case into the space, time and mass variation with velocity. This problem will be the one studied in the present section.

The usual relativistic space–time transformations can be written in the following terms, see, for example, [27]:

$${}^v x = \alpha(x - vt), \quad {}^v t = \alpha(t - c^{-1}\beta x), \quad (21)$$

where

$$\alpha = (1 - \beta^2)^{-1/2}, \quad \beta = \frac{v}{c},$$

and the usual meaning has to be associated with the employed symbols: space coordinates ${}^v x, x$, time coordinates ${}^v t, t$, velocity v , light velocity c . The structure of every term shall be modified accordingly to the arbitrary dimensional framework of the present (hyper-)matrix description.

The nature of the parameters α and β will be reconsidered first, at the light of the new (hyper-)matrix framework proposed here. As the velocity \mathbf{V} now is considered as a (hyper-)matrix as in the classical case, then the (hyper-)matrix

$$\mathbf{B} = c^{-1}\mathbf{V}$$

has to alternatively substitute the original parameter β . With respect to the parameter α , one can see that it can be substituted by the (hyper-)matrix

$$\mathbf{A} = (\mathbf{1} - \mathbf{B}^{[2]})^{[-1/2]},$$

obtained by means of appropriate IMP powers. Then, the Lorentz-like transformation formulae can be written into the (hyper-)matrix structure as

$$\begin{aligned} {}^v\mathbf{X} &= \mathbf{A} * (\mathbf{X} - \mathbf{V} * \mathbf{T}), \\ {}^v\mathbf{T} &= \mathbf{A} * (\mathbf{T} - c^{-1}\mathbf{B} * \mathbf{X}), \end{aligned} \tag{22}$$

where the (hyper-)matrices ${}^v\mathbf{X}$, \mathbf{X}^3 and ${}^v\mathbf{T}$, \mathbf{T} denote the space-like and time-like (hyper-)matrix object descriptors, respectively, and the IMP has been used in order to obtain equivalent expressions to the scalar equations (21). This formulation can be interpreted as if every (hyper-)matrix component suffers a Lorentz-like transformation. The light velocity can be taken as a scalar or may be transformed into a scalar (hyper-)matrix, by simply defining the corresponding light velocity (hyper-)matrix as $\mathbf{C} = c\mathbf{1}$. The usual special relativity definitions and derived properties can be translated to the present space-time description with essentially the same results, as can be easily tested.

But it can be interesting to see if the scalar classical measures discussed in the previous section still hold. Accepting these special relativity parallel formulas, the variation of length and time with velocity could also be obtained through the cosine manipulations already discussed in the previous section, dealing with classical space-time of arbitrary dimensions. Thus, in this way the space variation may be computed taking a space-like (hyper-)matrix object \mathbf{X}_0 as a reference.

So, taking all this into account, the following angle-like function can be evaluated:

$$\lambda = \gamma(\mathbf{X}; \mathbf{X}_0). \tag{23}$$

Then, one will need the space-like object description in the moving frame:

$$\mathbf{X} = \mathbf{A}^{[-1]} * {}^v\mathbf{X} + \mathbf{V} * \mathbf{T},$$

and the needed scalar products will be obtained in turn as

$$\langle \mathbf{X} * \mathbf{X}_0 \rangle = \langle (\mathbf{A}^{[-1]} * {}^v\mathbf{X} + \mathbf{V} * \mathbf{T}) * \mathbf{X}_0 \rangle \tag{24}$$

³ The space variable (hyper-)matrix \mathbf{X} , appearing in the first row of equation (22), has somewhat a complicated structure, as can be seen in [28, p. 11]. Just to mimic the first part of equation (20) as well as in order to preserve simplicity, it will be left as a simple position array.

and

$$\langle \mathbf{X} * \mathbf{X} \rangle = \langle (\mathbf{A}^{[-1]} * {}^v\mathbf{X} + \mathbf{V} * \mathbf{T})^{[2]} \rangle,$$

which is if the space-like vector may be considered normalized, that is: $\langle \mathbf{X} * \mathbf{X} \rangle = 1$, while admitting that no generality is lost also supposing the reference space-like object description normalized, or $\langle \mathbf{X}_0 * \mathbf{X}_0 \rangle = 1$.

Thus, the function (23) will be expressed by the corresponding quotient like the one of equation (17), which owing to the normalization of the involved vectors is expressed directly as equation (24):

$$\begin{aligned} \lambda &= \gamma(\mathbf{X}; \mathbf{X}_0) \\ &\equiv \langle \mathbf{A}^{[-1]} * ({}^v\mathbf{X} + \mathbf{V} * \mathbf{T}) * \mathbf{X}_0 \rangle \\ &= \langle \mathbf{A}^{[-1]} \rangle * \langle {}^v\mathbf{X} * \mathbf{X}_0 \rangle + \langle \mathbf{A}^{[-1]} * \mathbf{V} \rangle * \langle \mathbf{T} * \mathbf{X}_0 \rangle, \end{aligned}$$

where the asterisks enveloping the summation symbols denote Hadamard products [4]⁴. Then, in order to simplify the expression of the measure λ , one can also suppose the space-like reference representation almost orthogonal to the involved time-like representation, so it can be written:

$$\lambda \approx \langle \mathbf{A}^{[-1]} \rangle * \langle {}^v\mathbf{X} * \mathbf{X}_0 \rangle \equiv \langle \mathbf{A}^{[-1]} \rangle * \gamma({}^v\mathbf{X}; \mathbf{X}_0).$$

Thus, the following equation may be adopted at the end:

$$\lambda \approx \langle \mathbf{A}^{[-1]} \rangle * {}^v\lambda,$$

and the distance-like scalar $\ell = \lambda^{-1} - 1$ becomes in this case

$$\ell = (\langle \mathbf{A}^{[-1]} \rangle * {}^v\lambda)^{-1} - 1,$$

where some plausible simplification produces

$$\ell \approx \langle \mathbf{A} \rangle * {}^v\lambda^{-1} - 1 = \langle \mathbf{A} \rangle * ({}^v\ell + 1) - 1,$$

a result, which can be finally expressed simply as

$$\ell \approx \langle \mathbf{A} \rangle * {}^v\ell,$$

describing a variation of the distance-like scalar, associated to the space-like objects and measured in a cosine form. If this is feasible, then the observable object length will contract, as it must be expected from the usual special relativity result.

An equivalent reasoning may be followed for the time-like and mass-like descriptions, yielding equivalent results to the one-dimensional case, and for these reasons will not be repeated here.

⁴The *Hadamard product*, involving two summations bearing the same number of terms, is defined as yielding another sum made by the products of every pair of the initial sum elements. That is, generally speaking, $\langle \mathbf{A} \rangle * \langle \mathbf{B} \rangle = \sum_N(\mathbf{i}) a(\mathbf{i}) * \sum_N(\mathbf{i}) b(\mathbf{i}) = \sum_N(\mathbf{i}) a(\mathbf{i})b(\mathbf{i}) = \langle \mathbf{A} * \mathbf{B} \rangle$.

6. Conclusions and comments

6.1. Conclusions

With the aid of inward matrix product algebra, (hyper-)matrix representations of symmetrical space–time objects can be manipulated as in the scalar algebra physical counterparts. Interesting results, related to the Schrödinger equation and the associated matrix structure of mass and kinetic energy, as well as to the special relativity relationships can be obtained. A well-known measure concept, related to cosines of the angles subtended by two (hyper-)matrix objects, can be supposedly used to obtain the scalar values of the physical measures, and in this way the usual physical world measurement pictures can be easily connected with a superposed (hyper-)matrix description.

6.2. Comments

That time can be handled within quantum mechanics for each particle in a submicroscopic system in a non-scalar fashion, but in a manner similar to a space-like parameterisation, constitutes a subject, which has been already described and used several years ago in another context [29]. In this manner, one of the basic forms of the present study has somehow a particular precedent example.

In the same line of thought, but in a very different context, several space–time coordinates are used to describe turbulent flow [30]. In this way, the present treatment has not issued from the complete nothingness, but has a reasonable and well-documented background both at submicroscopic and macroscopic levels.

Thus, the plausible scenario, which the presented general theory contains, seems to point out towards a direction where space–time object descriptions can be easily generalised in the form of assigning to them some (hyper-)matrix characteristics, involving arbitrary dimensions. Experimental observations of some usual, positive definite, scalar physical magnitudes can be considered in such a situation the result of measurements made over multidimensional forms, which by the act of measure will yield such scalar values.

All of these theoretical speculations, which can be placed among several previous studies made by other authors [30–32], mainly dedicated to generalise the space-like features of the physical world, may indicate that, or such general point of view is just a mathematical exercise, extending the physical reality beyond the everyday experience, or due to the actual limitations, originated by our bio-physical structure, experimental space–time may possess an unobservable additional structure. Perhaps, at the light of the underlying thought of the present paper, a final question can now be set.

6.3. A final question

Owing to some limited nature of contemporary experimental possibilities, the present work may be, thus, resumed and at the same time ended, proposing the following question: if a crucial space–time super-structure like the one described in this

paper simply exists, will it just remain forever unattainable by experimental means, or alternatively will it be observable, albeit with extreme difficulty?

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